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# A Deterministic Inventory Replenishment Problem for Deteriorating Items with an Exponential Declining Demand and Shortages for the Finite Time Horizon

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**Abstract**

An inventory model is developed for deteriorating items with infinite rate of replenishment and exponentially decreasing demand rate over a finite time horizon. Shortages are allowed and backlogged. A numerical example demonstrates the effectiveness of the proposed method.

***Keywords:*** *Inventory replenishment, exponential declining demand, economic order quantity (EOQ).*

## 1. Introduction

Inventory problems involving time variable demand patterns have received the attention of several researchers in recent years. The classical no-shortage inventory problem for a linear trend in demand over a finite time-horizon was analytically solved by Donaldson [3]. Ritchie [6] obtained an exact solution, having the simplicity of the EOQ formula, for Donaldson's problem for linear, increasing demand. Dave and Patel [1] developed an inventory model for deteriorating items with time proportional demand. This model was extended by Sachan [7] to cover the backlogging option. Deb and Chaudhuri [2] studied the inventory replenishment policy for items having a deterministic demand patterns with a linear (positive) trend and shortages. This work was further extended by Murdeshwar[5].

Hollier and Mak [4] developed inventory replenishment policy for deteriorating items with a declining demand.

In the present paper, an inventory model is developed for deteriorating items with infinite rate of replenishment and exponentially decreasing demand rate over a finite time horizon. Further the shortages are allowed and excess demand is backlogged.

A numerical example is taken to show the effectiveness of the method. .

## 2. Assumptions and Notation

The proposed inventory model is developed under the following assumptions and notations:

(i) Replenishment rate is infinite.

(ii) The system operates for a prescribed period of H units of time (Planning horizon). Inventory level is zero at time t = 0 and at t=H.

(iii) Demand rate, D(t) is known and decreases exponentially, i.e., at time t, $t \geq 0$, $D(t) = Ae^{-\lambda t}$, A is initial demand and $\lambda$ is a constant governing the decreasing rate of the demand.

(iv) Lead time is zero.

(v) Shortages are allowed and are fully backlogged. Also shortages are not allowed in the final cycle.

(vi) $T_i$ is the total time that elapses upto and including the i-the cycle(i = =1,2,….., m), where, *m* denotes the total number of replenishment to be made during the prescribed time horizon H. Hence, $T_o = 0$ and $T_m = H$.

(vii) $t_i$ is the time at which the inventory level and the i-th cycle reaches zero, [i = 1,2,(m-1)].

(viii) T is the length of the first replenishment cycle and v is the rate of reduction of the successive cycle lengths.

(ix) The on hand inventory deteriorates at a constant rate of θ (0<θ<1) per time unit and there is no repair nor replacement of the deteriorated inventory during H.

(x) The inventory holding cost $C_1$ per unit per unit time, the shortage cost $C_2$ per unit per unit time, the unit cost C and the replenishment cost (ordering cost) $C_3$ per replenishment are known and constant during the planning time horizon H.

## 3. Mathematics Analysis

Let $Q_i(t)$ denote the amount in inventory at time *t* during the i-th cycle [$T_{i-1} \le t \le T_i$, i = 1,2,…., m)]. Then, the differential equations governing the system during i-th cycle are

$$\frac{dQ_i(t)}{dt} + \theta Q_i(t) = -Ae^{-\lambda t},$$

$T_{i-1} \le t \le t_i$, i = 1,2,…., m (3.1)

$$\frac{dQ_i(t)}{dt} = -Ae^{-\lambda t},$$

$t_i \le t \le T_i$, i = 1,2,…., m-1 (3.2)

subject to the conditions that *$Q_i(t) = 0$ at $t = t_i$*. The solution of (3.1) and (3.2) are

$$Q_i(t) = \frac{A}{\lambda - \theta}\left[e^{(\theta-\lambda)t_i - \theta t} - e^{-\lambda t}\right],$$

$T_{i-1} \le t \le t_i$, i = 1,2,…., m (3.3)

and,

$$Q_i(t) = \frac{A}{\lambda}\left[e^{\lambda t} - e^{-\lambda t_i}\right],$$

$t_i \le t \le T_i$, i = 1,2,…., m-1 (3.4)

Now the (i + 1)th the replenishment time $T_i$ can be expressed as

$T_i = iT - i(i-1)\,v/2$, i= 0,1,2,…(m-1) (3.5)

The length of the i-th cycle is
$T_i - T_{i-1} = T - (i-1)\,v$, i=1,2,……,m (3.6)
But the sum of the lengths of m replenishment cycles is H. Hence,

$$\sum_{i=1}^{m} [\,T - (i-1)v\,] = H \qquad \text{……(3.7)}$$

On simplifying the above relation, we have
$T = (m-1)\,v/2 + H/m$ …….(3.8)

The total cost of the system during the planning horizon H is

$$K(m, t_i, T_i) = mC_3 + \sum_{i=1}^{m-1}\left[(C_1+\theta C)\int_{T_{i-1}}^{t_i} Q_i(t)dt - C_2\int_{t_i}^{T_i} Q_i(t)dt\right] + (C_1+\theta C)\int_{T_{m-1}}^{H} Q_m(t)dt$$

$$K(m, t_i, T_i) = mC_3 + \sum_{i=1}^{m-1}\left[\frac{(C_1+\theta C)A}{\lambda-\theta}\int_{T_{i-1}}^{t_i}\left\{e^{(\theta-\lambda)t_i-\theta t} - e^{-\lambda t}\right\} - \frac{C_2 A}{\lambda}\int_{t_i}^{T_i}(e^{-\lambda t} - e^{-\lambda t_i})\right] + \frac{(C_1+\theta C)A}{\lambda-\theta}\int_{T_{m-1}}^{H}\left[e^{(\theta-\lambda)H-\theta t} - e^{-\lambda t}\right] \qquad (3.9)$$

Now for a fixed m, the corresponding optimal value of $t_i$ are the solutions of the system of (m-1) equations.

$$\frac{\partial K}{\partial t_i} = 0 \qquad [\, i = 1,2,……, (m-1)],$$

which implies that

$$t_i = \frac{(T_i - vT_{i-1})}{1-v}, \qquad i = 1, 2, \ldots, (m-1) \quad (3.10)$$

where, $v = \frac{C_1 + \theta C}{C_2}$.

The detailed calculation of (3.10*)* is given in Appendix A. Now simplifying (3.9), we have the cost function as: ……. (3.11)

$$K = mC_3 + \sum_{i=1}^{m-1}\left[\frac{(C_1+\theta C)A}{\lambda-\theta}\left[\frac{1}{\lambda}(e^{-\lambda t_i} - e^{-\lambda T_{i-1}}) + \frac{1}{\theta}e^{-\lambda t_i}(e^{\theta(t_i - T_{i-1})} - 1)\right] + C_2\left[\frac{A}{\lambda^2}(e^{-\lambda T_i} - e^{-\lambda t_i}) + \frac{A}{\lambda}e^{-\lambda t_i}(T_i - t_i)\right]\right] + \frac{(C_1+\theta C)A}{\lambda-\theta}\left[\frac{1}{\lambda}(e^{-\lambda H} - e^{-\lambda T_{m-1}}) + \frac{1}{\theta}e^{-\lambda H}(e^{\theta(H-T_{m-1})} - 1)\right] \quad (3.11)$$

While simplifying (3.9*)*, we have neglected $\theta^2$ and the higher power of θ. Since $0< \theta < 1$, using eqn.(3.10) in (3.11) we get on simplification,

$$K = mC_3 + \frac{(C_1+\theta C)A\lambda}{\lambda-\theta}\sum_{i=1}^{m-1}\left[\frac{(-T+(i-1)v)(T_i - vT_{i-1})}{(1-v)^2} + (T_{m-1} - H)H\right] - C_2A\sum_{i=1}^{m-1}\left[\frac{v(-T+(i-1)v)(T_i - vT_{i-1})}{(1-v)^2}\right] \quad (3.12)$$

Again substituting $T_i = iT - i(i-1)\,v/2$ and $T_i - T_{i-1} = T - (i-1)\,v$ in (3.12) and then simplifying, the cost function taken the form :

$$K = mC_3 + \frac{(C_1+\theta C)A\lambda}{(\lambda-\theta)(1-v)^2}\sum_{i=1}^{m-1}\{-T^2I_1 + I_4T(2I_1 - I_2) + I_5I_2 + I_3H(1-v)^2\} - \frac{C_2A}{(1-v)^2}\sum_{i=1}^{m-1}\{-T^2vI_1 + I_4vT(2I_1 - I_2) + I_5vI_2\} \quad (3.13)$$

where,

$I_1 = i - v(i-1)$,

$I_2 = v(i-2) - i$,

$I_3 = \{(m-1)T - (m-1)(m-2)v/2\} - H$,

$I_4 = (i-1)v/2$,

$I_5 = (i-1)^2 v^2/2$.

Substituting $T = (m-1)v/2 + H/m$ from (3.2) in (3.13) and also simplifying summation *i = 1 to m-1 for $I_1$, $I_2$, $I_3$, $I_4$* and $I_5$, the cost function K reduces to a function of two variables m and v only, of which m is a discrete and v is a continuous variable, and let it be K (m, v).

$$K(m,v) = mC_3 + \frac{(C_1+\theta C)A\lambda}{(\lambda-\theta)(1-v)^2}\left[-\frac{(m-1)^3mv^2}{8} + \frac{(m-1)^3(m-2)v^3}{8} - \frac{H^2}{m^2}\left\{\frac{(m-1)}{2}(m-(m-2)v)\right\} - \frac{(m-1)^2Hv}{2} + \frac{(m-1)^2(m-2)Hv^2}{2m} + \frac{m(m-1)^2(m-2)v^2}{4} - \frac{(m-1)^2(m-2)^2v^3}{4} + \frac{(m-1)(m-2)vH}{2}\right.$$

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$$\frac{(m-1)(m-2)^2Hv^2}{2m} + H(1-v)^2\left\{(m-1)((m-1)v/2+H/m) - \frac{(m-1)(m-2)v}{2} - H\right\}$$

+

$$\frac{m(m-1)}{24}\left(3m^2-19m+38-\frac{24}{m}\right)v^3 - \frac{m(m-1)(3m^2-11m+10)v^2}{24}\Bigg]$$

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$$\frac{C_2A}{(1-v)^2}$$

$$\Bigg[-\frac{(m-1)^3mv^3}{8}+\frac{(m-1)^3(m-2)v^4}{8}$$

-

$$\frac{H^2}{m^2}\left\{\frac{(m-1)v}{2}(m-(m-2)v)\right\}-\frac{(m-1)^2Hv^2}{2}+\frac{(m-1)^2(m-2)Hv^3}{2m}$$

+

$$\frac{m(m-1)^2(m-2)v^3}{4}+\frac{(m-1)(m-2)Hv^2}{2}-\frac{(m-1)^2(m-2)^2v^4}{4}$$

-

$$\frac{(m-1)(m-2)^2Hv^3}{2m}+\frac{m(m-1)}{24}\left(3m^2-19m+38-\frac{24}{m}\right)v^4$$

$$-\frac{m(m-1)(3m^2-11m+10)v^3}{24}\Bigg]$$ (3.14) For a given value $m_o$(> 1) of m, the optimal value of v for minimum total average cost is the solution of

$$\frac{dK(m_o,v)}{dv}=0$$ (3.15) provided that,

$$\frac{d^2K(m_o,v)}{dv^2}$$ > 0 for that value of v.

The optimal value of v (say, $v(m_o)$) can be obtained from 3.15 (the expression is very lengthy, so we avoid its presentation) by Newton - Raphson Method.

The corresponding optimal value of K is $\overline{K}$ ($m_o$, $v(m_o)$) (= K * ($m_o$)) which can be calculated from (3.13). Now putting $m_o$ = 2,3,4,…. one can easily calculateK*(2), K*(3), K*(4) …. .

For m=1, the system reduces to a single period and no shortage system with finite time horizon. In this case, the total cost is fixed and is given by, K*(1) = $C_3$ .

The minimum value of K*(1), K*(2),…..is the optimal cost and the corresponding value of $m_o$ (say m*) and v (say, v*) are their optimal values. The optimal values of T (say T*) and $T_i$ (say Ti*, i = 1,2,…,(m-1)) can be obtained from (3.8) and (3.5) respectively.
In this model, let θ → 0 (i.e. without deterioration), we obtain
$t_i = (T_i - v\,T_{i-1}) / (1 - v)$; i = 1,2,…, (m-1) (3.16)

where, $v = C_1/C_2$

The total cost for the entire horizon H can be obtained from (3.13) by substituting θ = 0 and is given by –

(K = $mC_3 + \frac{C_1A}{(1-v)^2}$ { $- T^2I_1 + I_4T\,(2I_1 – I_2) +$

$I_5I_2 + I_3H\,(1-v)^2$ }

$\frac{C_2A}{(1-v)^2}$ {$-T^2vI_1 + I_4vT\,(2I_1 – I_2) + I_5vI_2$ }(3.17)

## 4. Numerical Illustrations

The proposed model is illustrated with shortage by example with following data sets.
$C_1 = 1$, $C_2 = 3.5$, $C_3 = 10$, H = 0.01, λ = 0.30,
C = 5, A = 10

The optimum values of m, v, T and $T_i$ along with minimum total cost are calculated numerically for different values of θ.

## 5. Result

**Table 1**

| θ | 0.02 | 0.01 | 0.008 | 0.005 | 0.10 | 0.05 | 0.01 | 0.008 | 0.005 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| m* | 4 | 4 | 4 | 4 | 3 | 3 | 3 | 3 | 3 | 3 |
| V* | 0.2430 | 0.2219 | 0.2179 | 0.2120 | 0.5167 | 0.3170 | 0.2198 | 0.2157 | 0.2099 | 0.2004 |
| T* | 0.367 | 0.3353 | 0.3293 | 0.3205 | 0.5200 | 0.3203 | 0.2231 | 0.2190 | 0.2132 | 0.2037 |
| $T_i$* | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0.367 | 0.3353 | 0.3293 | 0.3205 | 0.5200 | 0.3203 | 0.2231 | 0.2190 | 0.2132 | 0.2037 |
| | 0.491 | 0.4487 | 0.4407 | 0.429 | 0.5233 | 0.3236 | 0.2264 | 0.2223 | 0.2165 | 0.207 |
| K* | 39.15 | 39.37 | 39.41 | 39.46 | 24.98 | 29.18 | 29.76 | 29.77 | 29.79 | 29.82 |

## 6. Discussion

A deterministic inventory model for deteriorating items with an exponential declining demand is developed for a fixed and finite planning horizon considering shortages.

It is observed, as per table that as deterioration decreases, the total cost increases. Also with the fall of number of replenishments, the total cost decreases. Results in the study can provide a valuable reference for decision-makers in planning the production and controlling the inventory.

This is because in practice, a situation arises commonly, when we observe the demand declining rapidly with time. For instance, the manufacturing of certain products may drop drastically due to introduction of new competitive products or changes in customer's preferences. Here, our model may provide a valuable reference for decision makers.

**Appendix – A**

Using (9) from $\frac{\partial K}{\partial t_i} = 0$ [ i = 1,2,….,(m-1)]

We have,

$$-- (C_1 + \theta C)A$$

$$.\int_{T_{i-1}}^{t_i} e^{(\theta-\lambda)t_i-\theta t}dt - C_2 A\int_{t_i}^{T_i} e^{-\lambda t_i}dt\ ,$$

or,

$$(C_1 + \theta C)$$

$$.\frac{A}{\theta}e^{-\lambda t_i}\left\{1 - e^{\theta(t_i - T_{i-1})}\right\} - C_2 A e^{-\lambda t_i}(T_i - t_i) \quad (A_1)$$

On solving, we get

$$(C_1 + \theta C)\ (T_{i-1} - t_i) - C_2\ (Ti - t_i) = 0$$

$$v\ (T_{i-1} - t_i) - (T_i - t_i) = 0$$

where, $v = .\frac{C_1 + \theta C}{C_2}$

$$v\,T_{i-1} - vt_i - T_i + t_i = 0$$

$$t_i\ (i - v) = T_i - v\,T_{i-1}$$

$$t_i = \frac{T_i - vT_{i-1}}{1 - v},\ i = 1,2,\ldots..\ (m-1)\ (A_2)$$

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